

Citations and the Zipf-Mandelbrot's law

Z. K. Silagadze

Budker Institute of Nuclear Physics, 630 090, Novosibirsk, Russia

Abstract

A curious observation was made that the rank statistics of scientific citation numbers follows Zipf-Mandelbrot's law. The same pow-like behavior is exhibited by some simple random citation models. The observed regularity indicates not so much the peculiar character of the underlying (complex) process, but more likely, than it is usually assumed, its more stochastic nature.

1 Introduction

Let us begin with an explanation as to what is Zipf's law. If we assign ranks to all words of some natural language according to their frequencies in some long text (for example the Bible), then the resulting frequency-rank distribution follows a very simple empirical law

$$f(r) = \frac{a}{r^\gamma} \quad (1)$$

with $a \approx 0.1$ and $\gamma \approx 1$. This was observed by G. K Zipf for many languages long time ago [1, 2]. More modern studies [3] also confirm a very good accuracy of this rather strange regularity.

In his attempt to derive the Zipf's law from the information theory, Mandelbrot [4, 5] produced a slightly generalized version of it:

$$f(r) = \frac{p_1}{(p_2 + r)^{p_3}}, \quad (2)$$

p_1, p_2, p_3 all being constants.

The same inverse pow-law statistical distributions were found in embarrassingly different situations (For reviews see [6, 7]). In economics, it was discovered by Pareto [8] long ago before Zipf and states that incomes of individuals or firms are inversely proportional to their rank. In less formal words [9], “most success seem to migrate to those people or companies who already are very popular”. In demography [2, 10, 11], city sizes (populations) also are pow-like functions of cities ranks. The same regularity reveals itself in the distributions of areas covered by satellite cities and villages around huge urban centers [12].

Remarkably enough, as is claimed in [13], in countries such as former USSR and China, where natural demographic process were significantly distorted, city sizes do not follow Zipf's law!

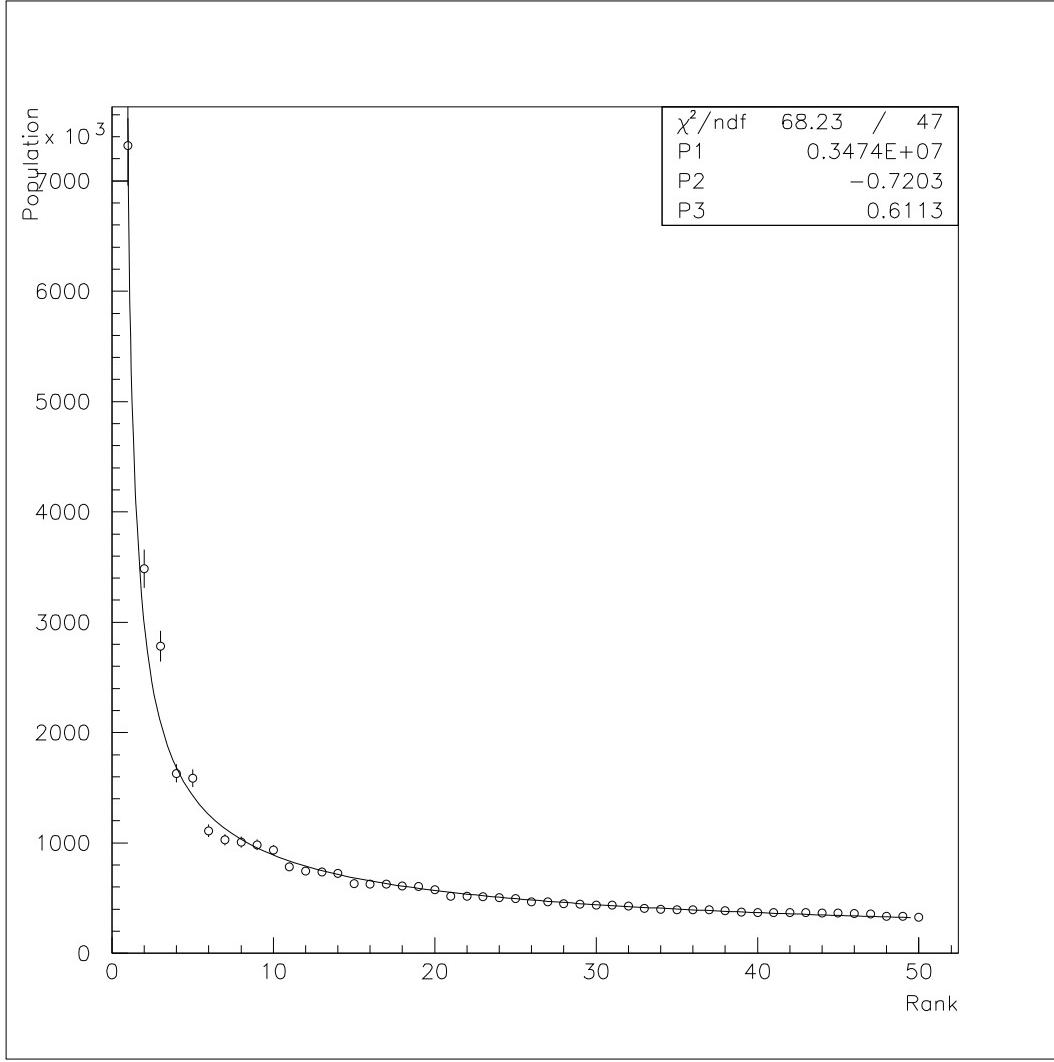
Other examples of zipfian behavior is encountered in chaotic dynamical systems with multiple attractors [14], in biology [15], ecology [16], social sciences and etc. [17].

Even the distribution of fundamental physical constants, according to [18], follows the inverse power law!

The most recent examples of Zipf-like distributions are related to the World Wide Web surfing process [19, 20].

You say that all this sounds like a joke and looks improbable? So did I when became aware of this weird law from M. Gell-Mann's book “The

Quark and the Jaguar” [21] some days ago. But here are the distribution of first 50 USA largest cities according to their rank [22], fitted by Eq.2:



The actual values of fitted parameters depend on the details of the fit. I assume (rather arbitrarily) 5% errors in data.

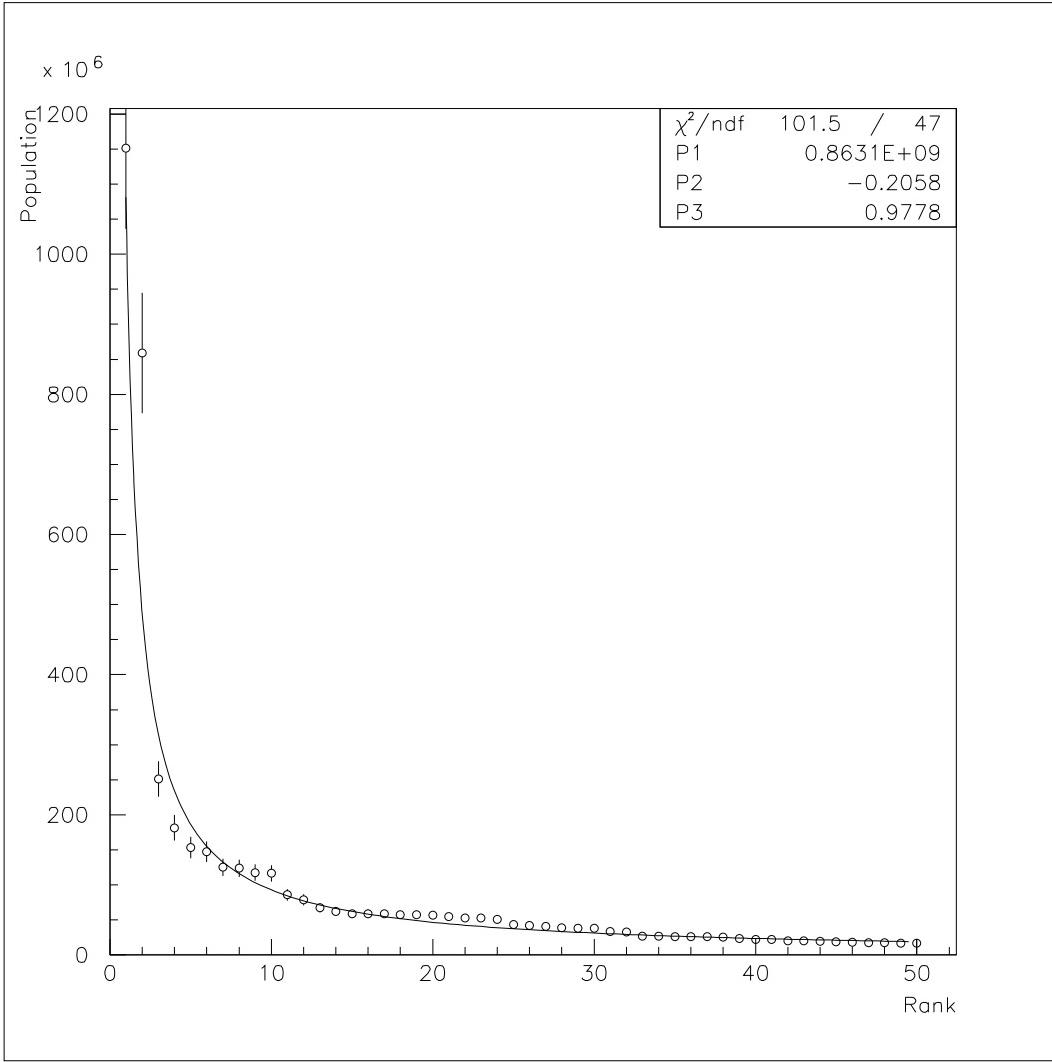
Maybe it is worthwhile to remember here, the old story about a young priest who complains his father about having a very difficult theme for his first public sermon – virgin birth.

– “Look father”, he says, “if some young girl from this town, becomes pregnant, comes to you and says that this is because of Holy Spirit. Do you believe it?”

The father stays silent for a while, then answers:

–”Yes, son, I do. If the baby would be born, if he would be raised and if he would live like the Christ”.

So, clearly, you need more empirical evidence to accept improbable things. Here is one more, the list of the most populated countries [23] fitted by the Mandelbrot formula (2):

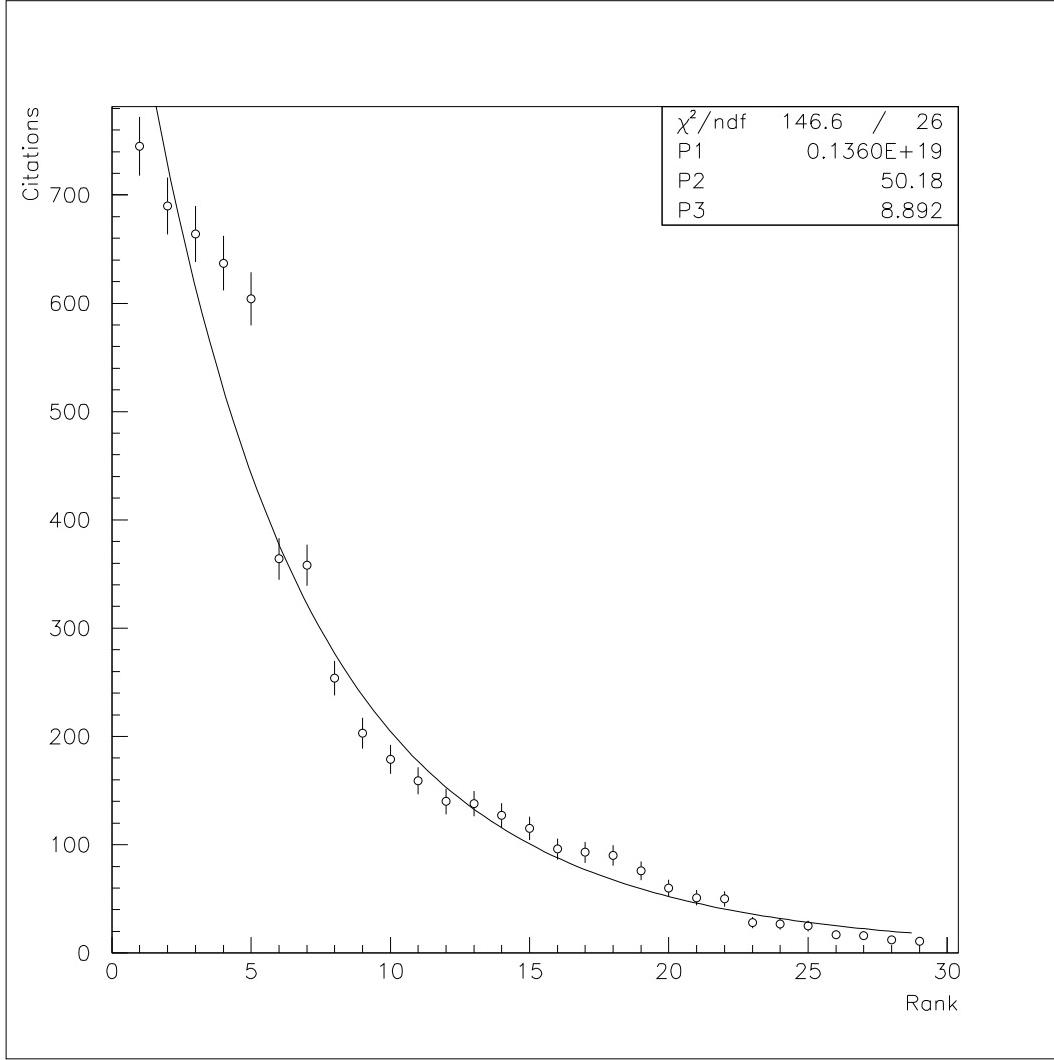


Even more simple Zipfian a/r parameterization will work in this case fairly well!

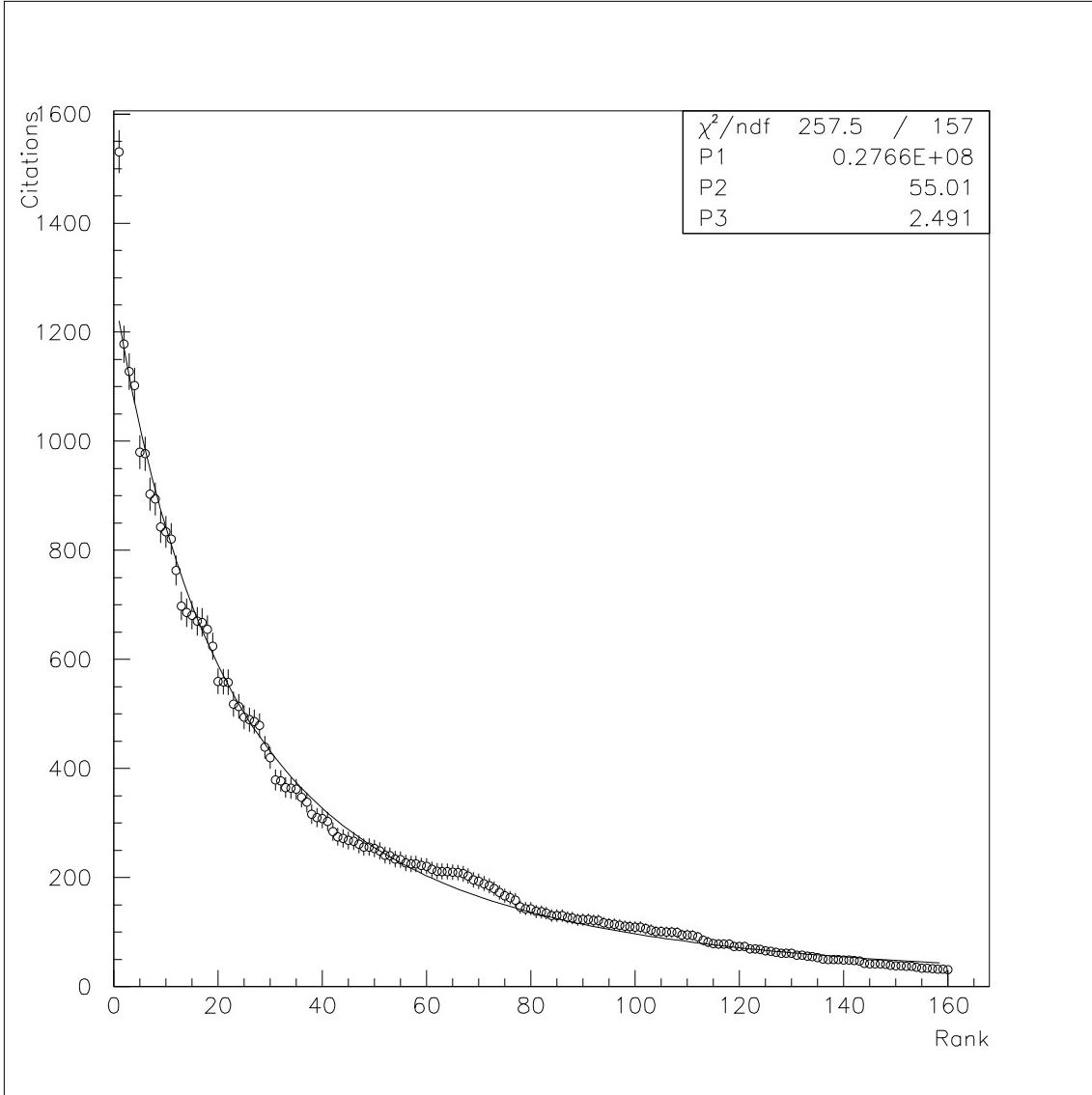
2 Fun with citations

But all this was known long ago. Of course it is exciting to check its correctness personally. But more exciting is to find whether this rule still holds in a new area. SPIRES database provides excellent possibility to check scientific citations against Zipf-Malderbrot's regularity.

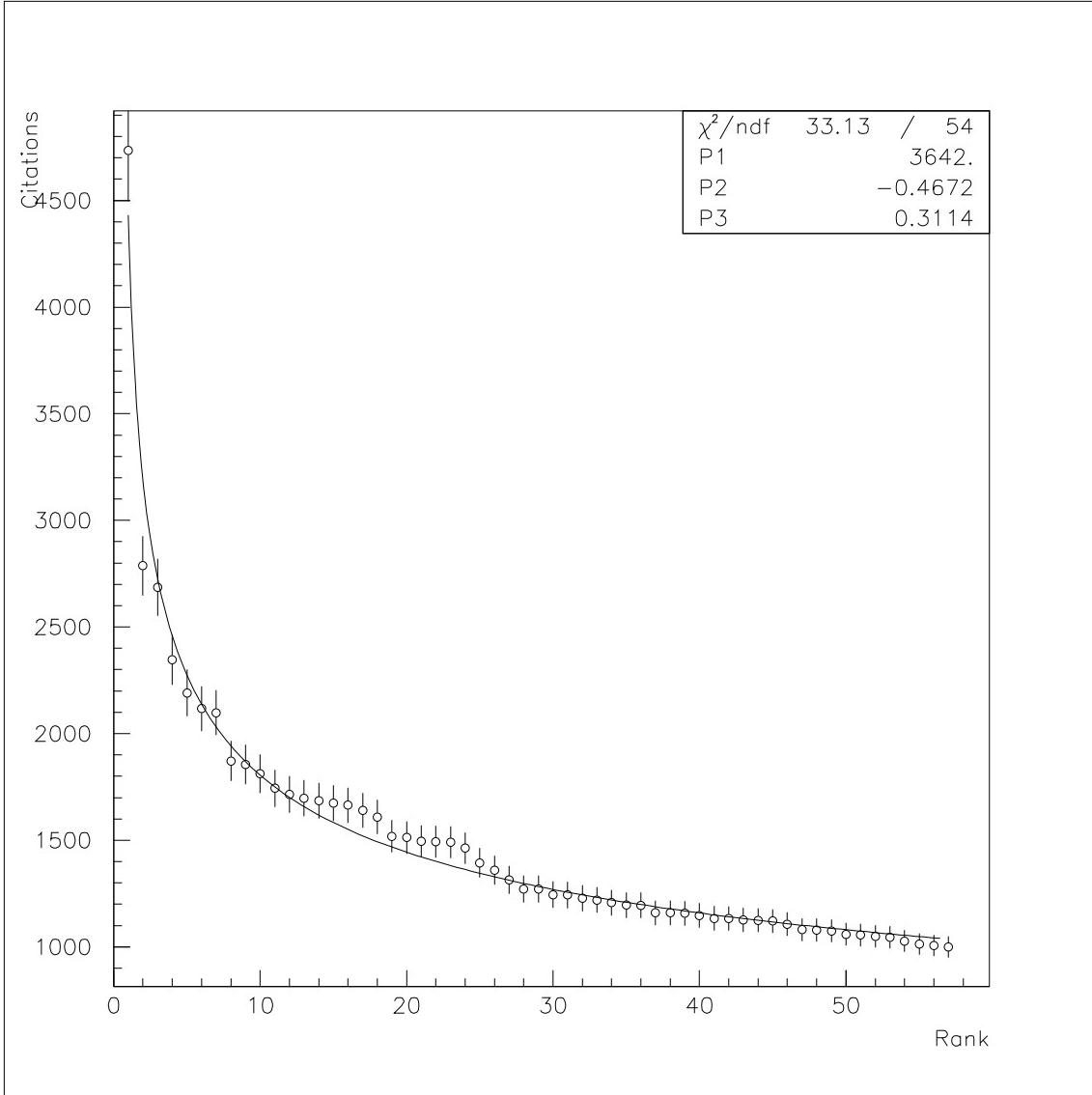
As I have been involved in this matters because of M. Gell-Mann's book, my first try naturally was his citations itself. The results were encouraging:



But maybe M. Gell-Mann is not the best choice for this goal. SPIRES is a rather novel phenomenon, and M. Gell-Mann's many important papers were written long before its creation. So they are purely represented in the database. Therefore, let us try present day citation favorite E. Witten. Here are his 160 most cited papers according to SPIRES [24] (Note once more that the values of fitted parameters may depend significantly on the details of the fit. In this and previous case I choose \sqrt{N} as an estimate for data errors, not to ascribe too much importance to data points with small numbers of citations. In other occasions I assume 5% errors. Needless to say, both choices are arbitrary):

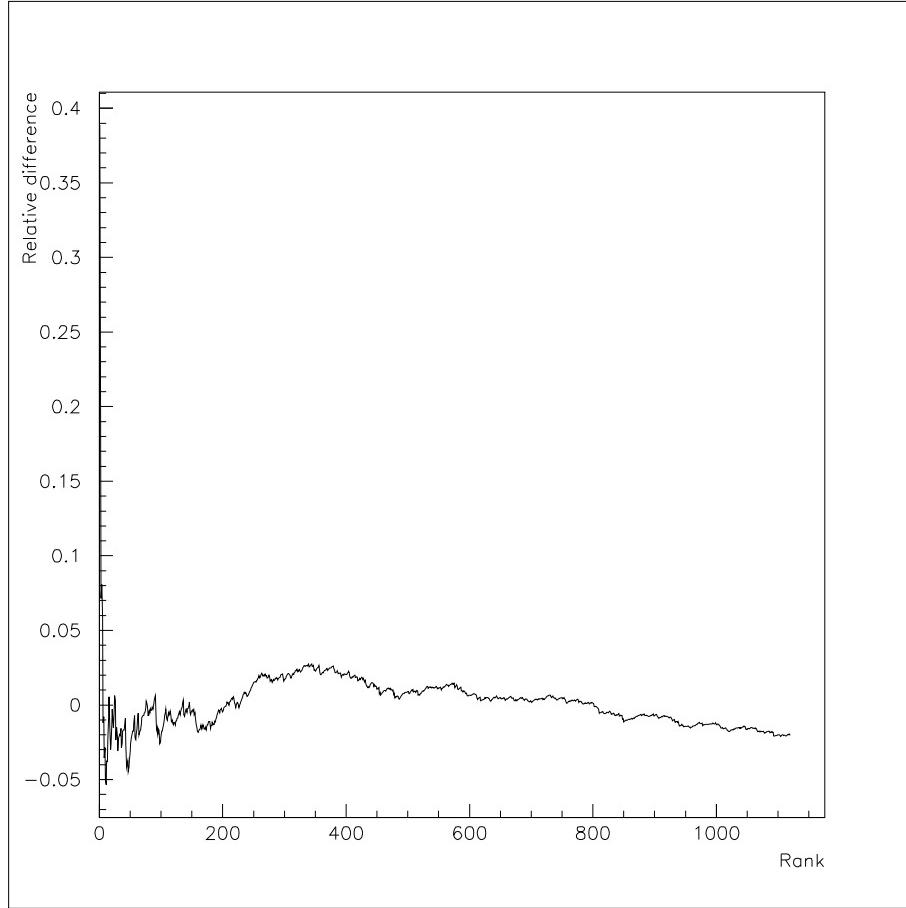


You have probably noticed very big values of the prefactor p_1 . Of course this is related to the rather big values of other two parameters. We can understand big value of p_2 parameter as follows. The data set of individual physicist's papers are subset of more full data about all physicists. So we can think of p_2 as being an average number of papers from other scientists between two given papers of the physicists under consideration. Whether right or not, this explanation gains some empirical support if we consider top cited papers in SPIRES [25] (Review of particle physics is excluded):



As we see p_2 is fairly small now.

At last, it is possible to find the list of 1120 most cited physicists (not only from the High Energy Physics) on the World Wide Web [26]. Again the Mandelbrot formula (2) with $p_1 = 3.81 \cdot 10^4$, $p_2 = 10.7$ and $p_3 = 0.395$ gives an excellent fit. Now there are too many points, making it difficult to note visually the differences between the curve and data. In the figure that follows, we show this relative difference explicitly.



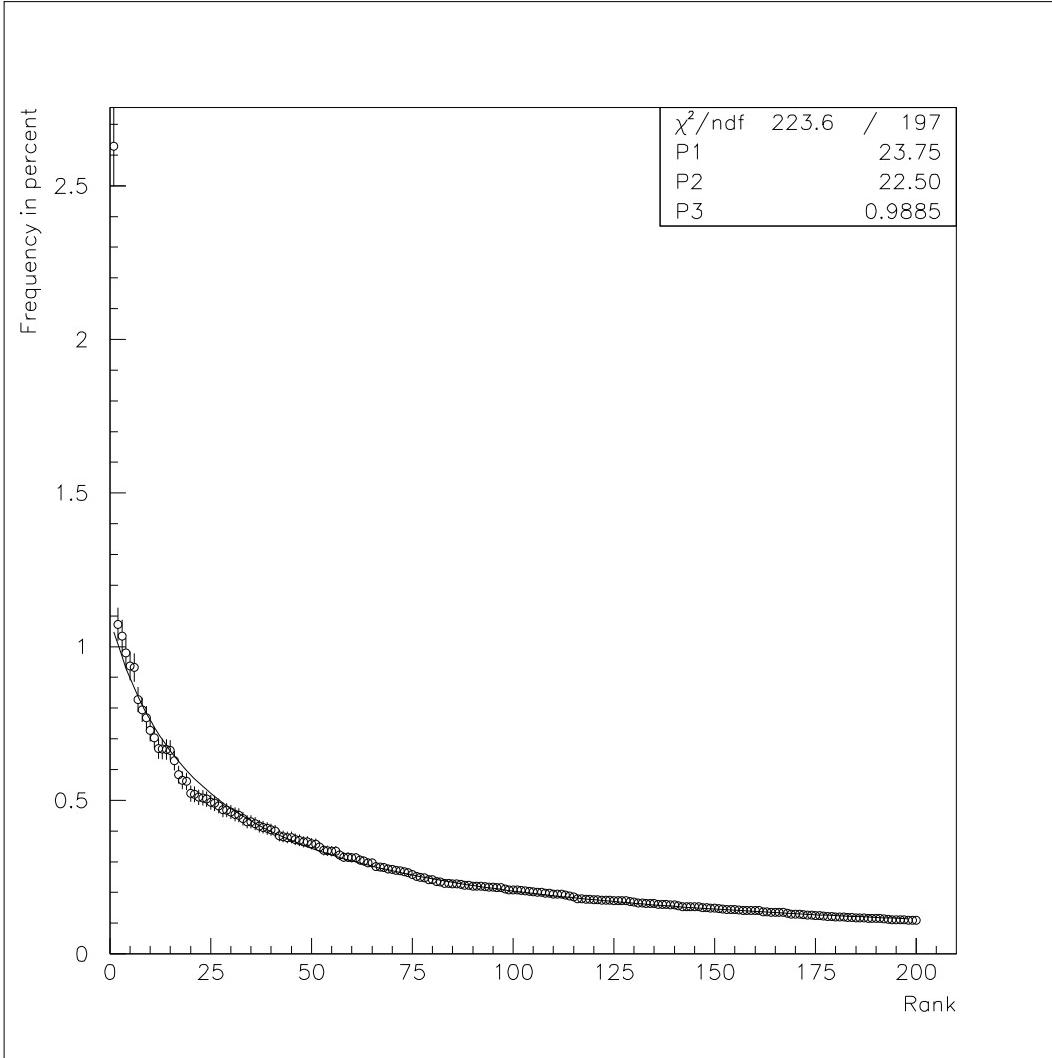
For the most bulk of data the Mandelbrot's curve gives the precision better than 5%!

You wonder why now p_2 is relatively high? I really do not know. Maybe the list is still incomplete for his lower rank part. In any case, if you take just the first 100 entries from this list, the fit results in $p_1 = 2.1 \cdot 10^4$, $p_2 = -0.09$, $p_3 = 0.271$. This example also shows that actually the Mandelbrot's curve with constant p_1 , p_2 , p_3 is not as good approximation as one might judge from the above given histograms, because different parts of data prefer different values of the Mandelbrot's parameters.

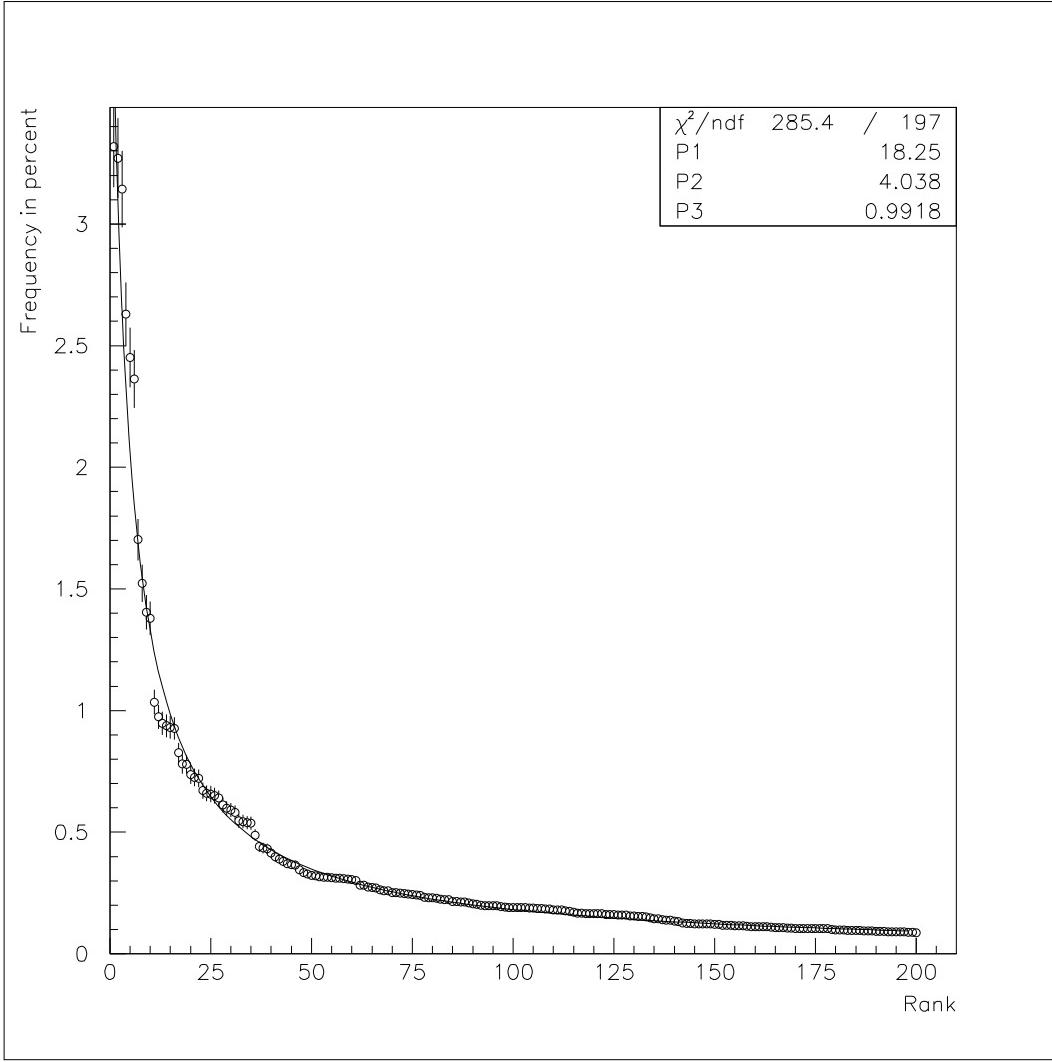
3 Any explanation?

The general character of the Zipf-Mandelbrot's law is hypnotizing. We already mentioned several wildly different areas where it was encountered. Can it be considered as some universal law for complex systems? And if so, what is the underlying principle which unifies all of these seemingly different

systems? What kind of principle can be common for natural languages, individual wealth distribution in some society, urban development, scientific citations, and female first name frequencies distribution? The latter is reproduced below [27]:



Another question is whether the Mandelbrot's parameters p_2 and p_3 can tell us something about the (complex) process which triggered the corresponding Zipf-Mandelbrot distribution. For this goal an important issue is how to perform the fit (least square, χ^2 , method of moments [20] or something else?). I do not have any answer to this question now. However let us compare the parameters for the female first name distribution from the above given histogram and for the male first name distribution (data are taken from the same source [27]). In both cases χ^2 fit was applied with 5% errors assumed for each point.



The power-counting parameter p_3 is the same for both distributions, although the p_2 parameter has different values.

If you are fascinated by a possibility that very different complex systems can be described by a single simple law, you maybe will be disappointed (as was I) to learn that some simple stochastic processes can lead to very same Zipfian behavior. Say, what profit will you have from knowing that some text exhibits Zipf's regularity, if this gives you no idea the text was written by Shakespeare or by monkey? Alas, it was shown [4, 28, 29, 30] that random texts ("monkey languages") exhibit Zipf's-law-like word frequency distribution. So Zipf's law seems to be at least [5] "linguistically very shallow" and [29] "is not a deep law in natural language as one might first have thought".

Two different approaches to the explanation of Zipf's law is very well

summarized in G. Millers introduction to the 1965 edition of Zipf’s book [1]: “Faced with this massive statistical regularity, you have two alternatives. Either you can assume that it reflects some universal property of human mind, or you can assume that it reflects some necessary consequence of the laws of probabilities. Zipf chose the synthetic hypothesis and searched for a principle of least effort that would explain the apparent equilibrium between uniformity and diversity in our use of words. Most others who were subsequently attracted to the problems chose the analytic hypothesis and searched for a probabilistic explanation. Now, thirty years later, it seems clear that the others were right. Zipf’s curves are merely one way to express a necessary consequence of regarding a message source as a stochastic process”.

Were “others” indeed right? Even in the realm of linguistics the debate is still not over after another thirty years have passed [31]. In the case of random texts, the origin of the Zipf’s law is well understood [32, 33]. In fact such texts exhibit no Zipfian distribution at all, but log-normal distribution, the latter giving in some cases a very good approximation to the Zipf’s law. So there is no doubt that simple stochastic (Bernoulli or Markov) processes can lead to a Zipfian behavior. No dynamically non-trivial properties (interactions and interdependence) is required at all from the underlying system. But it was also stressed in the literature [34, 13] that this fact does not preclude more complex and realistic systems to exhibit Zipfian behavior because of underlying nontrivial dynamics. In this case, we can hope that the Zipf-Mandelbrot parameters will be meaningful and can tell something about the system properties. Let us note that the rank-frequency distribution for complex systems is not always Zipfian. For example, if we consider the frequency of occurrence of letters, instead of words, in a long text, the empirical universal behavior, valid over 100 natural languages with alphabet sizes ranged between 14 and 60, is logarithmic [35]

$$f(r) = A - B \ln r$$

where A and B are constants. This fact, of course, is interesting by itself. It is argued in [35] that both regularities (zipfian and logarithmic) can have the common stochastic origin.

An interesting example of Zipf-Mandelbrot’s parameters being useful and effective, is provided by ecology [36, 37]. The exponent p_3 is related to the evenness of the ecological community. It has higher values for “simple” and lower values for “complex” systems. The parameter p_2 is related to

the “diversity of the environment” [37] and serves as a measure of the complexity of initial preconditions.

The another pole in explanation of Zipf’s law seeks some universal principle behind it, such as “least effort” [2], “minimum cost” [4], “minimum energy” [38] or “equilibrium” [39]. The most impressive and, as the above ecological example shows, fruitful explanation is given by B. Mandelbrot [5, 40] and is based on fractals and self-similarity.

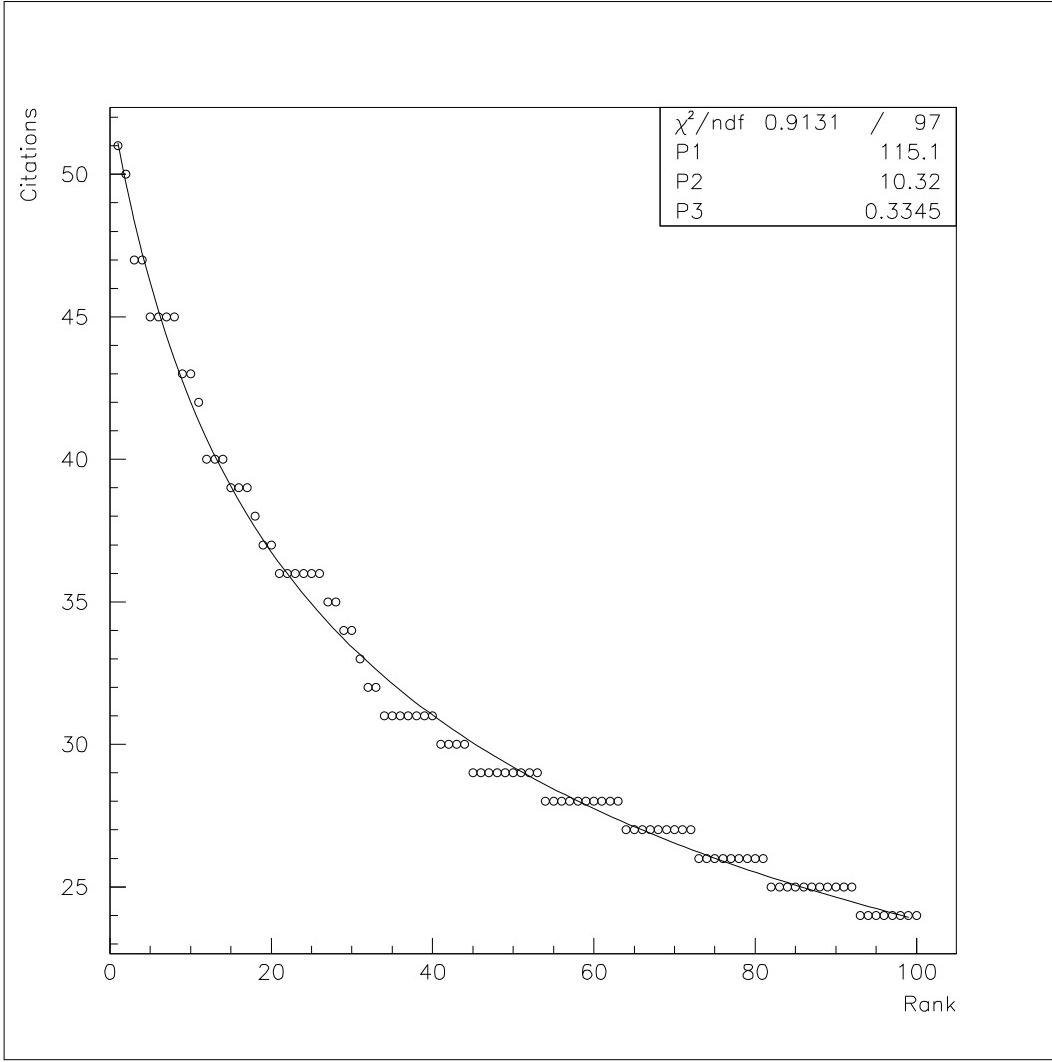
As we see, the suggested explanations are almost as numerous as the observed manifestations of this universal pow-like behavior. This probably indicates that some important ingredient in this regularity still escapes to be grasped. As M. Gell-Mann concludes [21] “Zipf’s law remains essentially unexplained”.

4 The almighty chance

If monkeys can write texts they can make citations too! So let us imagine the following random citation model.

- At the beginning there is one “seminal” paper.
- Every sequential paper makes at most ten citations (or cites all preceding papers if their number does not exceed ten).
- All preceding papers have an equal probability to be cited.
- Multiple citations are excluded. So if some paper is selected by chance as an citation candidate more than once, the selection is ignored (in this case total number of citations in a new paper will be less than ten).

I doubt about monkeys but it is simple to learn computer to simulate such a process. Here is the result of simulation for 1000 papers.



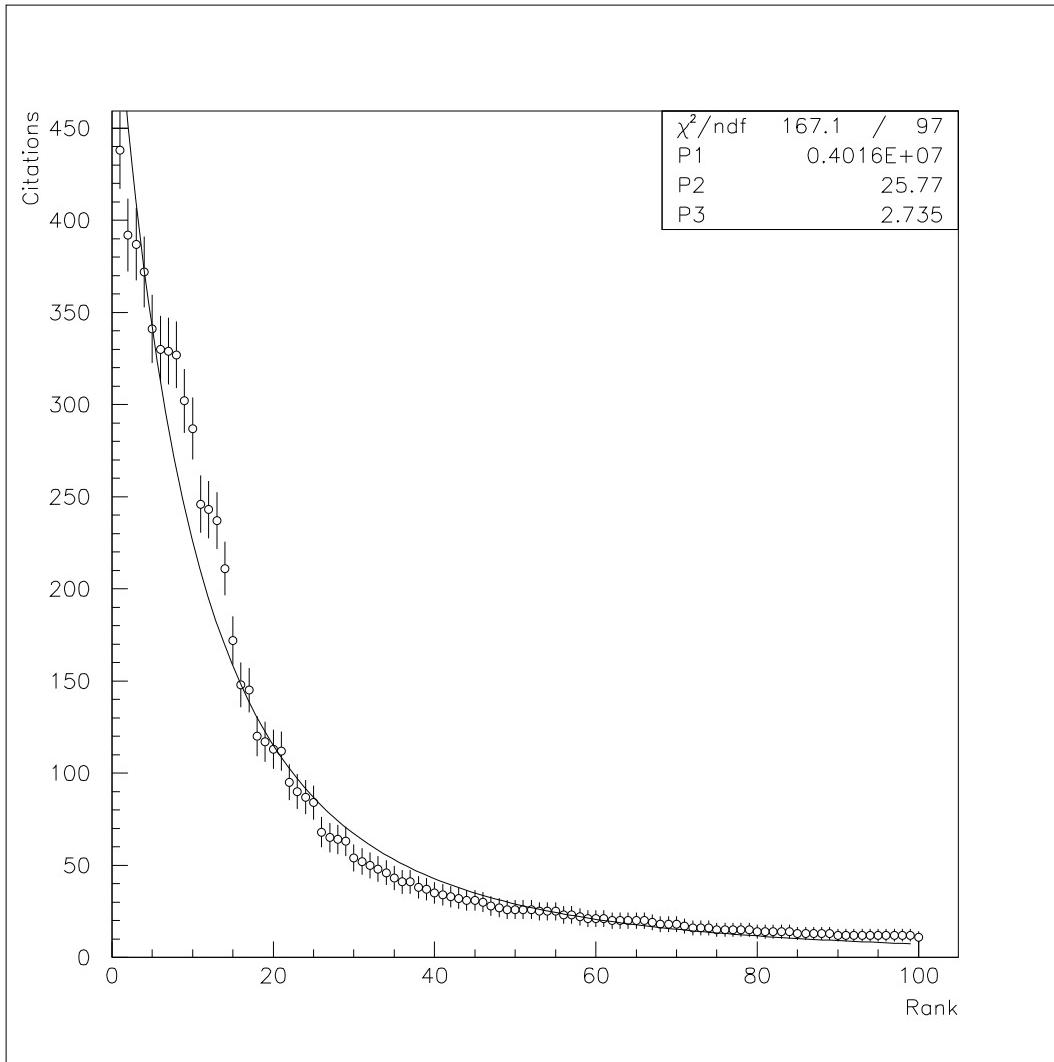
So we see an apparent pow-like structure, although with staircase behavior. We expect this stepwise structure to disappear if we eliminate the democracy between papers and make some papers more probable to be cited.

Note that even the value of exponent p_3 is reasonably close to what was really observed for the most cited papers. But this can be merely an accident and I do not like to make some farfetched conclusion about the nature of citation process from this fact.

In reality “Success seems to attract success” [9]. Therefore, let us try to see what happens if the equal probability axiom is changed by perhaps a more realistic one:

- The probability for a paper to be cited is proportional to $n + 1$, where n is the present total citation number for the paper.

It is still assumed that all preceding papers compete to be cited by a new paper, but with probabilities as follows from the above given law. The result for 1000 papers now looks like



The fit seems not so good now, nevertheless you can notice some resemblance with the case of individual scientists. Again I refrain from premature conclusions. Although it is not entirely surprising that the well-known a given paper of a certain author is, the more probable becomes its citation in a new paper.

5 Discussion

So scientific citations (leaving aside first name frequencies) provides one more example of Zipf-Mandelbrot's regularity. I do not know whether this

fact indicates only to significant stochastic nature of the process or to something else. In any case SPIRES, and the World Wide Web in general, gives us an excellent opportunity to study the characteristics of the complex process of scientific citations.

I do not know either whether Mandelbrot's parameters are meaningful in this case, and if they can tell us something non-trivial about the citation process.

The very generality of the Zipf-Mandelbrot's regularity can make it rather "shallow". But remember, that the originality of answers on the question of whether there is something serious behind the Zipf-Mandelbrot's law depends how restrictive frameworks we assume for the answer. Shallow framework will probably guarantee shallow answers. But if we do not restrict our imagination from the beginning, answers can turn out to be quite non-trivial. For example, fractals and self-similarity are certainly great and not shallow ideas. This point is very well illustrated by the "Barometer Story", which I like so much that I'm tempted to reproduce it here (it is reproduced as given in M. Gell-Mann's book [21]).

6 The Barometer Story – by Dr. A. Calandra

Some time ago, I received a call from a colleague who asked if I would be the referee on the grading of an examination question. It seemed that he was about to give a student a zero for his answer to a physics question, while the student claimed he should receive a perfect score and would do so if the system were not set up against the student. The instructor and the student agreed to submit this to an impartial arbiter, and I was selected...

I went to my colleague's office and read the examination question, which was, "Show how it is possible to determine the height of a tall building with the aid of a barometer."

The student's answer was, "Take the barometer to the top of the building, attach a long rope to it, lower the barometer to the street, and then bring it up, measuring the length of the rope. The length of the rope is the height of the building."

Now this is a very interesting answer, but should the student get credit for it? I pointed out that the student really had a strong case for full credit, since he had answered the question completely and correctly. On the other hand, if full credit were given, it could well contribute to a high grade for the student in his physics course. A high grade is supposed to certify that

the student knows some physics, but the answer to the question did not confirm this. With this in mind, I suggested that the student have another try at answering the question. I was not surprised that my colleague agreed to this, but I was surprised that the student did.

Acting in the terms of the agreement, I gave the student six minutes to answer the question, with the warning that the answer should show some knowledge of physics. At the end of five minutes, he had not written anything. I asked if he wished to give up, since I had another class to take care of, but he said no, he was not giving up, he had many answers to this problem, he was just thinking of the best one. I excused myself for interrupting him to please go on. In the next minute, he dashed off his answer, which was: “Take the barometer to the top of the building, and lean over the edge of the roof. Drop the barometer, timing its fall with a stopwatch. Then, using the formula $s = at^2/2$, calculate the height of the building.”

At this point, I asked my colleague if he would give up. He conceded and I gave the student almost full credit. In leaving my colleague’s office, I recalled that the student had said that he had other answers to the problem, so I asked him what they were.

“Oh, yes,” said the student. “There are many ways of getting the height of a tall building with the aid of a barometer. For example, you could take the barometer out on a sunny day and measure the height of the barometer, the length of its shadow, and the length of the shadow of the building, and by the use of simple proportion, determine the height of the building.”

“Fine,” I said. “And the others?”

“Yes”, said the student. “There is a very basic measurement that you will like. In this method, you take the barometer and begin to walk up the stairs. As you climb the stairs, you mark off the length and this will give you the height of the building in barometer units. A very direct method.”

“Of course, if you want a more sophisticated method, you can tie the barometer to the end of a string, swing it as a pendulum, and determine the value of g at the street level and at the top of the building. From the difference between the two values of g , the height of the building can, in principle, be calculated.”

Finally, he concluded, “If you don’t limit me to physics solution to this problem, there are many other answers, such as taking the barometer to the basement and knocking on the superintendent’s door. When the superintendent answers, you speak to him as follows:

Dear Mr. Superintendent, here I have a very fine barometer. If you will tell me the height of this building, I will give you this barometer ...”

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Note added

After this paper was completed and submitted to e-Print Archive, I have learned that the Zipf’s distribution in scientific citations was discovered in fact earlier by S. Redner [41]. He also cites some previous studies on citations, which were unknown to me.

I also became aware of G. Parisi’s interesting contribution [42] from Dr. S. Juhos.

I thank S. Redner and S. Juhos for their correspondence.

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